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# Three coefficients of a polynomial can determine its $\phi$ -instability

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## Abstract

Let  $\phi \in [\frac{\pi}{2}, \pi)$ . A polynomial  $P(x) = \sum_{i=0}^n a_i x^{n-i}$  with real positive coefficients is said to be  $\phi$ -stable if any root  $re^{i\theta}$  of  $P(x)$  satisfies that  $r > 0$  and  $\theta \in (\phi, 2\pi - \phi)$ . We will see that in certain cases it is enough to know three coefficients of  $P(x)$  in order to conclude that  $P(x)$  is  $\phi$ -unstable.

The case  $\phi = \frac{\pi}{2}$  was considered in [A. Borobia, S. Dormido, Three coefficients of a polynomial can determine its instability, *Linear Algebra Appl.* 338 (2001) 67–76] (note that  $\frac{\pi}{2}$ -stability is Hurwitz stability). Now assume that  $\phi \in (\frac{\pi}{2}, \pi)$ , that  $k$  is an integer with  $0 < k < n$  and that we know the coefficients  $a_0$ ,  $a_k$  and  $a_n$  of  $P(x)$ . We will calculate a positive number  $\gamma = \gamma(\phi, n, k, a_0, a_n)$  with the following property: if  $a_k \leq \gamma$  then  $P(x)$  is  $\phi$ -unstable, and if  $a_k > \gamma$  then  $P(x)$  is  $\phi$ -stable or  $\phi$ -unstable depending on the rest of its coefficients.

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## 1. Introduction

A real polynomial is said to be a *stable* or a *Hurwitz* polynomial if and only if all its roots lie in the open left half of the complex plane. Stable polynomials are important in the roots location problem for polynomials. The study of roots location of polynomials has a tradition in the applied mathematics that goes back almost two centuries (see Marden [8]). It is a fact that most of the

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initial results were concerned with only a single polynomial rather than a family of polynomials. Indeed, the problem to determine if a given single real polynomial is stable was completely solved by the criterion provided by Hurwitz [3].

A great effort has been made in the robust formulation of the problem, that concerns the roots location for families of polynomials whose coefficients depend on parameters. There is an extensive Control Theory literature on this particular subject that goes back at least half a century. The parametric space methods developed in the 1960s and 1970s are prime examples of work in this direction (see [1]). In this sense, the seminal theorem of Kharitonov [5] in 1978 regarding the Hurwitz stability of a family of interval polynomials has been the takeoff point for much of the research in this area.

Note that in this context the Hurwitz's criterion is not so useful since it can not be efficiently applied to each single polynomial of a given family. Therefore any new stability condition that can be easily tested is of the utmost interest.

The simplest and well known necessary condition for the stability of a real polynomial is that all coefficients of the polynomial have the same sign. Based on this property we will restrict ourselves to the study of polynomials with positive coefficients.

Hurwitz stability is generalized as follows. Let  $\phi \in [\frac{\pi}{2}, \pi)$ . A polynomial  $P(x)$  with real positive coefficients is said to be  $\phi$ -stable if any root  $re^{i\theta}$  of  $P(x)$  satisfies that  $r > 0$  and  $\theta \in (\phi, 2\pi - \phi)$ . Note that  $\frac{\pi}{2}$ -stability is Hurwitz stability.

The solution to the problem to determine if a given single real polynomial is  $\phi$ -stable can be found in Jury [4]. But again we are interested in the analysis of roots location for families of polynomials whose coefficients depend on parameters. As stated above, any new  $\phi$ -stability condition that can be easily tested is of the utmost interest. In this context, we will see that three coefficients of  $P(x)$  can determine its  $\phi$ -instability.

Previous results concerning criteria for the  $\phi$ -stability of a real polynomial can be found in works by Lipatov [7], Nemirovskii and Polyak [9].

## 2. The results in [2]

The results in [2] complement those by Nemirovskii and Polyak [9]. Such a kind of results were previously obtained by Krueger [6].

Let  $\mathbf{P}_n$  be the set of polynomials of order  $n$  with positive coefficients, and for  $\phi \in [\frac{\pi}{2}, \pi)$  let  $\mathbf{SP}_n^\phi$  denote the set of  $\phi$ -stable polynomials of order  $n$ . For any  $t$  integers  $i_1, \dots, i_t$  with  $0 \leq i_1 < \dots < i_t \leq n$  and any  $t$  positive numbers  $\delta_1, \dots, \delta_t > 0$  define the set

$$\mathbf{P}_n[(a_{i_1}, \delta_1), \dots, (a_{i_t}, \delta_t)] = \left\{ \sum_{i=0}^n a_i x^{n-i} \in \mathbf{P}_n : a_{i_1} = \delta_1, \dots, a_{i_t} = \delta_t \right\}.$$

Define also the set

$$\mathbf{SP}_n^\phi[(a_{i_1}, \delta_1), \dots, (a_{i_t}, \delta_t)] = \mathbf{P}_n[(a_{i_1}, \delta_1), \dots, (a_{i_t}, \delta_t)] \cap \mathbf{SP}_n^\phi.$$

The knowledge of two coefficients of a polynomial of  $\mathbf{P}_n$  is not sufficient in order to conclude that it is  $\frac{\pi}{2}$ -unstable.

**Theorem 2.1.**  $\mathbf{SP}_n^{\pi/2}[(a_i, \alpha), (a_j, \beta)] \neq \emptyset$  for any  $\alpha, \beta > 0$ .

The situation changes when we know three coefficients.

**Theorem 2.2.**  $\text{SP}_n^{\pi/2}[(a_0, 1), (a_s, \delta), (a_n, 1)] = \emptyset$  if and only if  $s$  and  $n$  are even numbers and  $\delta \leq \binom{n/2}{s/2}$ .

Theorem 2.2 was widely generalized by Yang [10,11].

### 3. Symmetric functions

Let  $n_1, \dots, n_p \in \mathbb{N}$  with  $n_1 + \dots + n_p = n$ , and let  $k_1, \dots, k_p \in \mathbb{N} \cup \{0\}$  with  $k_1 > \dots > k_p \geq 0$ . Define  $\mathbf{Z}_n[n_1 : k_1, \dots, n_p : k_p]$  as the set

$$\{\mathbf{z} = (z_1, \dots, z_n) : \mathbf{z} \text{ has } n_i \text{ coordinates equal to } k_i \text{ for } i = 1, \dots, p\}.$$

Note that

$$|\mathbf{Z}_n[n_1 : k_1, \dots, n_p : k_p]| = \frac{n!}{n_1! \cdots n_p!}.$$

For  $t > 0$  and  $\varepsilon \geq 0$  define the set

$$\mathbf{H}_{n,t^n}^\varepsilon = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1, \dots, x_n \geq \varepsilon, \quad x_1 \cdots x_n = t^n\}$$

and define the symmetric function

$$\begin{aligned} \Gamma_{n,t^n}^\varepsilon[n_1 : k_1, \dots, n_p : k_p] : \mathbf{H}_{n,t^n}^\varepsilon &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\mapsto \sum_{\mathbf{z} \in \mathbf{Z}_n[n_1 : k_1, \dots, n_p : k_p]} x_1^{z_1} \cdots x_n^{z_n}. \end{aligned}$$

In Lemma 3.1 we will write in the language that we have just introduced the well known relation between the coefficients of a polynomial and its roots. Lemma 3.2 is a useful result of the same nature and its proof is straightforward. Finally, Corollary 3.1 is a very particular case of Lemma 3.2.

**Lemma 3.1.** *Let*

$$P(x) = \prod_{i=1}^n (x + r_i) = \sum_{i=0}^n A_i x^{n-i},$$

*then*

$$A_i = \Gamma_{n, \prod_{i=1}^n r_i}^0[i : 1, n - i : 0](r_1, \dots, r_n).$$

**Lemma 3.2.** *Let*

$$P(x) = \prod_{i=1}^n (x^2 + \beta r_i x + r_i^2) = \sum_{i=0}^{2n} B_i x^{2n-i}.$$

*If  $c = 0$  or  $1$  and  $0 \leq 2q + c \leq 2n$ , then*

$$B_{2q+c} = \sum_{g=0}^q \beta^{2g+c} \Gamma_{n, \prod_{i=1}^n r_i}^0[q - g : 2, 2g + c : 1, n - q - g - c : 0](r_1, \dots, r_n).$$

**Corollary 3.1.** *Let*

$$P(x) = (x^2 + \beta x + 1)^n = \sum_{i=0}^{2n} C_{2n,i}^\beta x^{2n-i}.$$

If  $c = 0$  or  $1$  and  $0 \leq 2q + c \leq 2n$ , then

$$C_{2n, 2q+c}^{\beta} = \sum_{g=0}^q \beta^{2g+c} \frac{n!}{(q-g)!(2g+c)!(n-q-g-c)!}.$$

#### 4. Main result

A first trivial result states that if we know two coefficients of a polynomial of  $\mathbf{P}_n$  we cannot conclude that it is  $\phi$ -unstable. In order to simplify its proof we introduce a new definition. For any real polynomial  $P(x) = \sum_{k=0}^n a_k x^{n-k}$  and any  $\lambda, \mu \in \mathbb{R}$  define

$$P(x, \lambda, \mu) = \sum_{k=0}^n \lambda \mu^k a_k x^{n-k}.$$

Note that  $z$  is a root of  $P(x)$  if and only if  $\mu z$  is a root of  $P(x, \lambda, \mu)$ . In particular, if  $\mu > 0$  then  $P(x) \in \mathbf{SP}_n^{\phi}$  if and only if  $P(x, \lambda, \mu) \in \mathbf{SP}_n^{\phi}$ .

**Lemma 4.1.**  $\mathbf{SP}_n^{\phi}[(a_i, \alpha), (a_j, \beta)] \neq \emptyset$  for any  $\phi \in [\frac{\pi}{2}, \pi)$  and any  $\alpha, \beta > 0$ .

**Proof.** Consider any  $P(x) = \sum_{i=0}^n a_i x^{n-i} \in \mathbf{SP}_n^{\phi}$  and let  $\mu, \lambda > 0$  be given by

$$\mu = \left( \frac{\alpha a_j}{\beta a_i} \right)^{\frac{1}{i-j}} \quad \text{and} \quad \lambda = \frac{\beta}{\mu^j a_j}.$$

An immediate conclusion is that  $P(x, \lambda, \mu) \in \mathbf{SP}_n^{\phi}$ . Indeed, it can be directly checked that  $P(x, \lambda, \mu) \in \mathbf{SP}_n^{\phi}[(a_i, \alpha), (a_j, \beta)]$ .  $\square$

The situation changes when we know three coefficients of a polynomial of  $\mathbf{P}_n$ . It is important to advise that we will restrict our study to the set  $\mathbf{SP}_n^{\phi}[(a_0, 1), (a_n, 1)]$ . A good reason for this restriction is that for any  $P(x) = \sum_{i=0}^n a_i x^{n-i} \in \mathbf{P}_n$  we have that

$$P(x) \in \mathbf{SP}_n^{\phi} \Leftrightarrow P\left(x, \frac{1}{a_0}, \left(\frac{a_0}{a_n}\right)^{\frac{1}{n}}\right) \in \mathbf{SP}_n^{\phi}[(a_0, 1), (a_n, 1)].$$

Moreover, to fix the first and the last coefficients has advantages with calculations.

Now we can state our main result (recall the value of  $C_{2n,s}^{\beta}$  given in Corollary 3.1).

**Theorem 4.1.** Let  $\phi \in [\frac{\pi}{2}, \pi)$  and  $\beta = -2\cos\phi$ . Then

- (i)  $\mathbf{SP}_{2n}^{\phi}[(a_0, 1), (a_s, \delta), (a_{2n}, 1)] = \emptyset$  if and only if  $\delta \leq C_{2n,s}^{\beta}$ .
- (ii)  $\mathbf{SP}_{2n+1}^{\phi}[(a_0, 1), (a_s, \delta), (a_{2n+1}, 1)] = \emptyset$  if and only if

$$\delta \leq (2n+1) \left( \frac{C_{2n,s-1}^{\beta}}{s} \right)^{\frac{s}{2n+1}} \left( \frac{C_{2n,s}^{\beta}}{2n+1-s} \right)^{1-\frac{s}{2n+1}}.$$

**Remark.** For  $\phi = \frac{\pi}{2}$  we have  $\beta = 0$  and therefore

$$C_{2n,2s}^0 = \binom{n}{s} \quad \text{and} \quad C_{2n,2s+1}^0 = 0,$$

which implies that Theorem 2.2 is a very particular case of Theorem 4.1.

## 5. Auxiliary result

In this section we will calculate the extreme values of  $\Gamma_{n,t^n}^\varepsilon[n_1 : k_1, \dots, n_p : k_p]$ , and we will calculate the points of  $\mathbf{H}_{n,t^n}^\varepsilon$  where that extreme values are reached.

Note that if  $\varepsilon > t$  then  $\mathbf{H}_{n,t^n}^\varepsilon = \emptyset$ , and if  $\varepsilon = t$  then  $\mathbf{H}_{n,t^n}^\varepsilon = \{(t, \dots, t)\}$ . In order to avoid trivial situations we will suppose that  $0 \leq \varepsilon < t$ . Now we can state the main result of this section.

**Lemma 5.1.** *Let  $n_1, \dots, n_p \in \mathbb{N}$  with  $n_1 + \dots + n_p = n$ , let  $k_1, \dots, k_p \in \mathbb{N} \cup \{0\}$  with  $k_1 > \dots > k_p \geq 0$ , let  $N = n_1 k_1 + \dots + n_p k_p$ , and let  $\varepsilon, t \in \mathbb{R}$  with  $0 \leq \varepsilon < t$ . The following equalities and inequalities are satisfied:*

(I) *If  $p = 1$  then  $\Gamma_{n,t^n}^\varepsilon[n : k](\mathbf{x}) = t^{nk}$  for all  $\mathbf{x} \in \mathbf{H}_{n,t^n}^\varepsilon$ .*

(II) *If  $p \geq 2$  then*

$$\Gamma_{n,t^n}^\varepsilon[n_1 : k_1, \dots, n_p : k_p](\mathbf{x}) \geq \frac{n!}{n_1! \dots n_p!} t^N \quad (1)$$

*and equality holds only if  $\mathbf{x} = (t, \dots, t)$ .*

(III) *If  $\varepsilon > 0$  then*

$$\Gamma_{n,t^n}^\varepsilon[n_1 : k_1, \dots, n_p : k_p](\mathbf{x}) \leq \sum_{i=1}^p \frac{(n-1)!}{n_1! \dots n_{i-1}! (n_i-1)! n_{i+1}! \dots n_p!} t^{nk_i} \varepsilon^{N-nk_i} \quad (2)$$

*and equality holds only in the  $n$  points of  $\mathbf{H}_{n,t^n}^\varepsilon$  with  $n-1$  coordinates equal to  $\varepsilon$ .*

**Proof.** (I) It is trivial.

(II) Proof of inequality (1): In [2] we proved inequality (1) for the very particular case of  $\Gamma_{n,1}^0[r : 1, n-r : 0]$ . Here we will use the same line of argument, although calculations are more complicated.

Note that for each  $0 \leq \varepsilon < t$  we have that  $(t, \dots, t) \in \mathbf{H}_{n,t^n}^\varepsilon$ , that  $\mathbf{H}_{n,t^n}^\varepsilon \subset \mathbf{H}_{n,t^n}^0$ , and that for each  $\mathbf{x} \in \mathbf{H}_{n,t^n}^\varepsilon$

$$\Gamma_{n,t^n}^\varepsilon[n_1 : k_1, \dots, n_p : k_p](\mathbf{x}) = \Gamma_{n,t^n}^0[n_1 : k_1, \dots, n_p : k_p](\mathbf{x}).$$

Therefore, if we prove the result for  $\mathbf{H}_{n,t^n}^0$  it will follow for all  $\mathbf{H}_{n,t^n}^\varepsilon$ . We divide this proof in four steps:

(i) For each  $i, j \in \{1, \dots, n\}$  with  $i \neq j$  and each  $\lambda \in \mathbb{R} - \{0\}$  define the function

$$T_{i,j}^\lambda : \begin{array}{ccc} \mathbf{H}_{n,t^n}^0 & \longrightarrow & \mathbf{H}_{n,t^n}^0 \\ (a_1, \dots, a_n) & \mapsto & (b_1, \dots, b_n) \end{array}$$

with  $b_i = \lambda a_i$ ,  $b_j = \frac{a_j}{\lambda}$  and  $b_k = a_k$  if  $k \neq i, j$ . If  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{H}_{n,t^n}^0$  with  $a_i \geq a_j$  and  $\lambda > 1$  then

$$\Gamma_{n,t^n}^0[n_1 : k_1, \dots, n_p : k_p](T_{i,j}^\lambda(\mathbf{a})) - \Gamma_{n,t^n}^0[n_1 : k_1, \dots, n_p : k_p](\mathbf{a})$$

is equal to

$$\sum_{\substack{k_u, k_v \in \{k_1, \dots, k_p\} \\ k_u > k_v}} A(k_u, k_v) \cdot B(k_u, k_v) \cdot C(k_u, k_v),$$

where

$$\begin{aligned} A(k_u, k_v) &= a_i^{k_v} a_j^{k_v} \quad \text{and} \\ B(k_u, k_v) &= \left( (\lambda a_i)^{k_u - k_v} + \left( \frac{a_j}{\lambda} \right)^{k_u - k_v} \right) - \left( a_i^{k_u - k_v} + a_j^{k_u - k_v} \right) \\ &= (\lambda^{k_u - k_v} - 1) \left( a_i^{k_u - k_v} - \left( \frac{a_j}{\lambda} \right)^{k_u - k_v} \right). \end{aligned}$$

We can assume without loss of generality that  $i < j$ . Let  $n'_u = n_u - 1$ ,  $n'_v = n_v - 1$ , and  $n'_k = n_k$  for  $k \in \{1, \dots, n\}$  and  $k \neq u, v$ . Then

$$C(k_u, k_v) = \sum_{\mathbf{z} \in \mathbf{Z}_{n-2}[n'_1:k_1, \dots, n'_p:k_p]} a_1^{z_1} \cdots a_{i-1}^{z_{i-1}} a_{i+1}^{z_{i+1}} \cdots a_{j-1}^{z_{j-1}} a_{j+1}^{z_{j+1}} \cdots a_n^{z_{n-2}}.$$

It is not difficult to see that  $A(k_u, k_v), B(k_u, k_v), C(k_u, k_v) > 0$  and therefore

$$\Gamma_{n,t^n}^0[n_1 : k_1, \dots, n_p : k_p](T_{i,j}^\lambda(\mathbf{a})) > \Gamma_{n,t^n}^0[n_1 : k_1, \dots, n_p : k_p](\mathbf{a}).$$

(ii) Let  $(a_1, \dots, a_n), (c_1, \dots, c_n) \in \mathbf{H}_{n,t^n}^0$  be such that it is satisfied for each  $k \in \{1, \dots, n\}$  that if  $c_k \geq t$  then  $c_k \geq a_k \geq t$ , and if  $c_k \leq t$  then  $c_k \leq a_k \leq t$ . Suppose that there exists some  $i \in \{1, \dots, n\}$  such that  $c_i > a_i \geq t$ , then also there exists some  $j \in \{1, \dots, n\} - \{i\}$  such that  $c_j < a_j \leq t$ . Let  $\beta = \min \left\{ \frac{c_i}{a_i}, \frac{a_j}{c_j} \right\} > 1$  and

$$(b_1, \dots, b_n) = T_{i,j}^\beta(a_1, \dots, a_n).$$

Then  $c_i \geq b_i > a_i \geq t$  and  $c_j \leq b_j < a_j \leq t$ , and for  $h \neq i, j$   $b_h = a_h$ . Therefore, it is satisfied for each  $k \in \{1, \dots, n\}$  that if  $c_k > t$  then  $c_k \geq b_k \geq a_k \geq t$  and if  $c_k < t$  then  $c_k \leq b_k \leq a_k \leq t$ . Moreover  $(c_1, \dots, c_n)$  and  $(b_1, \dots, b_n)$  have 1 or 2 more common coordinates than  $(c_1, \dots, c_n)$  and  $(a_1, \dots, a_n)$  since  $c_i = b_i$  or  $c_j = b_j$ , or both coincide.

(iii) From (ii) it follows that for any  $(b_1, \dots, b_n) \in \mathbf{H}_{n,t^n}^0$  there exists a finite sequence of at most  $n$  elements

$$(a_1^{(0)}, \dots, a_n^{(0)}), (a_1^{(1)}, \dots, a_n^{(1)}), \dots, (a_1^{(r)}, \dots, a_n^{(r)}) \in \mathbf{H}_{n,t^n}^0$$

that satisfies:

1.  $(a_1^{(0)}, \dots, a_n^{(0)}) = (t, \dots, t)$  and  $(a_1^{(r)}, \dots, a_n^{(r)}) = (b_1, \dots, b_n)$ .
2. For  $k \in \{1, \dots, n\}$  if  $b_k \geq t$  then

$$b_k = a_k^{(r)} \geq a_k^{(r-1)} \geq \dots \geq a_k^{(0)} = t.$$

3. For  $k \in \{1, \dots, n\}$  if  $b_k \leq t$  then

$$b_k = a_k^{(r)} \leq a_k^{(r-1)} \leq \dots \leq a_k^{(0)} = t.$$

4. For  $h \in \{0, 1, \dots, r-1\}$  the points  $(a_1^{(h+1)}, \dots, a_n^{(h+1)})$  and  $(b_1, \dots, b_n)$  have 1 or 2 more common coordinates than the points  $(a_1^{(h)}, \dots, a_n^{(h)})$  and  $(b_1, \dots, b_n)$ .

Now we explain how to construct the sequence. Suppose  $(a_1^{(h)}, \dots, a_n^{(h)})$  is different from  $(b_1, \dots, b_n)$ . Then there exists some  $i_h \in \{1, \dots, n\}$  such that  $b_{i_h} > a_{i_h}^{(h)} \geq t$ , and some  $j_h \in \{1, \dots, n\} - \{i_h\}$  with  $b_{j_h} < a_{j_h}^{(h)} \leq t$ . Take

$$\lambda_h = \min \left\{ \frac{b_{i_h}}{a_{i_h}^{(h)}}, \frac{a_{j_h}^{(h)}}{b_{j_h}} \right\} > 1$$

and define

$$(a_1^{(h+1)}, \dots, a_n^{(h+1)}) = T_{i_h, j_h}^{\lambda_h} (a_1^{(h)}, \dots, a_n^{(h)}).$$

(iv) From (i) and (iii) it follows that

$$\Gamma_{n, t^n}^0 [n_1 : k_1, \dots, n_p : k_p](t, \dots, t) = \frac{n!}{n_1! \dots n_p!} t^N$$

is smaller than  $\Gamma_{n, t^n}^0 [n_1 : k_1, \dots, n_p : k_p](\mathbf{x})$  for all  $\mathbf{x} \in \mathbf{H}_{n, t^n}^0 - \{(t, \dots, t)\}$ .

(III) Proof of inequality (2): Let  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{H}_{n, t^n}^\varepsilon$ . We can assume without loss of generality that coordinates are ordered in such a way that  $a_1 \geq a_i \forall i = 2, \dots, n$ . Note that  $a_1 \leq t_n \varepsilon^{1-n}$  and  $a_i \geq \varepsilon \forall i = 2, \dots, n$ . Let  $\lambda_i = \frac{a_i}{\varepsilon} \geq 1 \quad \forall i = 2, \dots, n$ . Then,

$$T_{1, n}^{\lambda_n} \left( T_{1, n-1}^{\lambda_{n-1}} \left( \dots \left( T_{1, 2}^{\lambda_2}(\mathbf{a}) \right) \dots \right) \right) = (t^n \varepsilon^{1-n}, \varepsilon, \dots, \varepsilon).$$

Arguing as in (II) (i) we obtain that

$$\Gamma_{n, t^n}^\varepsilon [n_1 : k_1, \dots, n_p : k_p](\mathbf{a}) \leq \Gamma_{n, t^n}^\varepsilon [n_1 : k_1, \dots, n_p : k_p](t^n \varepsilon^{1-n}, \varepsilon, \dots, \varepsilon)$$

and equality is reached only when  $\mathbf{a} = (t^n \varepsilon^{1-n}, \varepsilon, \dots, \varepsilon)$  since in this case we have that  $\lambda_2 = \dots = \lambda_n = 1$ . The value of  $\Gamma_{n, t^n}^\varepsilon [n_1 : k_1, \dots, n_p : k_p](t^n \varepsilon^{1-n}, \varepsilon, \dots, \varepsilon)$  is

$$\sum_{i=1}^p \varepsilon^{N-k_i} \frac{(n-1)!}{n_1! \dots n_{i-1}! (n_i-1)! n_{i+1}! \dots n_p!} (t^n \varepsilon^{1-n})^{k_i}$$

and the result follows.  $\square$

## 6. Proof of Theorem 4.1

(A) Let  $\phi \in [\frac{\pi}{2}, \pi)$ ,  $d = 0$  or  $1$ , and

$$P(x) = \sum_{i=0}^{2n+d} a_i x^{2n+d-i} \in \mathbf{SP}_{2n+d}^\phi [(a_0, 1), (a_{2n+d}, 1)].$$

Let

$$-t_1, \dots, -t_{2g+d} \in \mathbb{R} \quad \text{and} \\ z_1 = r_1 e^{i\theta_1}, \quad \bar{z}_1, \dots, z_h = r_h e^{i\theta_h}, \quad \bar{z}_h \in \mathbb{C} - \mathbb{R}$$

be the roots of  $P(x)$ . Note that

$$\prod_{i=1}^{2g+d} t_i \prod_{j=1}^h r_j^2 = 1 \quad \text{and} \quad 2g + d + 2h = 2n + d.$$

As the polynomial is  $\phi$ -stable then  $t_i > 0$  for  $i = 1, \dots, 2g + d$ . Moreover, we can take  $r_j > 0$  and  $\phi < \theta_j < \pi$  for  $j = 1, \dots, h$ . Let  $\gamma_j = -2 \cos \theta_j > 0$ . Consider the following sequence of polynomials:

1. Define  $P_1(x) = P(x)$ , that is,

$$\begin{aligned} P_1(x) &= \prod_{i=1}^{2g+d} (x + t_i) \prod_{j=1}^h ((x - z_j)(x - \bar{z}_j)) \\ &= \prod_{i=1}^{2g+d} (x + t_i) \prod_{j=1}^h (x^2 + \gamma_j r_j x + r_j^2). \end{aligned}$$

2. Let  $t^{2g+d} = \prod_{i=1}^{2g+d} t_i$ , define

$$P_2(x) = (x + t)^{2g+d} \prod_{j=1}^h (x^2 + \gamma_j r_j x + r_j^2).$$

3. Let  $\beta = -2 \cos \phi > 0$ , define

$$P_3(x) = (x + t)^d (x^2 + \beta t x + t^2)^g \prod_{j=1}^h (x^2 + \beta r_j x + r_j^2).$$

Define  $r_{h+1} = r_{h+2} = \dots = r_{h+g} = t$ , then  $t^d \prod_{j=1}^n r_j^2 = 1$  and

$$P_3(x) = (x + t)^d \prod_{j=1}^n (x^2 + \beta r_j x + r_j^2).$$

4. Let  $r > 0$  such that  $r^{2n} = t^{-d} = \prod_{j=1}^n r_j^2$ , define

$$P_4(x) = (x + t)^d (x^2 + \beta r x + r^2)^n.$$

(B) We will prove that

$$P_1(x) \geq P_2(x) \geq P_3(x) \geq P_4(x),$$

where  $P_i(x) \geq P_j(x)$  means that  $P_i(x) - P_j(x)$  has nonnegative coefficients.

(i)  $P_1(x) \geq P_2(x)$ : We must to show that

$$\prod_{i=1}^{2g+d} (x + t_i) \geq (x + t)^{2g+d}.$$

The result follows from Lemma 3.1 and Lemma 5.1.

(ii)  $P_2(x) \geq P_3(x)$ : This is obvious since  $\beta < 2$  and  $\beta < \gamma_j$  for all  $j = 1, \dots, h$ . Moreover, as we have strict inequalities for  $\beta$  then it will follow that the coefficient of  $x^i$  in  $P_2(x)$  is greater than the coefficient of  $x^i$  in  $P_3(x)$  for  $i = 1, \dots, 2n + d - 1$ . Note that for  $i = 0$  or  $2n + d$  the coefficients of  $x^i$  in  $P_2(x)$  and  $P_3(x)$  are equal to 1.

(iii)  $P_3(x) \geq P_4(x)$ : We must to show that



$$\prod_{j=1}^n (x^2 + \beta r_j x + r_j^2) \geq (x^2 + \beta r x + r^2)^n.$$

The result follows from Lemma 3.2 and Lemma 5.1.

(C) We will distinguish two cases depending on the value of  $d$ :

(i) Case  $d = 0$ : Then  $P_4(x) = (x^2 + \beta x + 1)^n$ . As we saw in Corollary 3.1

$$(x^2 + \beta x + 1)^n = \sum_{i=0}^{2n} C_{2n,i}^{\beta} x^{2n-i}.$$

Note that in (A) we have started with an arbitrary polynomial

$$P(x) = \sum_{i=0}^{2n} a_i x^{2n-i} \in \mathbf{SP}_{2n,\phi}[(a_0, 1), (a_{2n}, 1)],$$

and in (B) we have showed that  $P(x) \geq (x^2 + \beta x + 1)^n$ . Then it follows that  $a_r \geq C_{2n,r}^{\beta}$ . On the other hand, by the argument given in (B) (ii) we conclude that the inequality is strict.

It remains to prove that

$$\delta > C_{2n,s}^{\beta} \Rightarrow \mathbf{SP}_{2n}^{\phi}[(a_0, 1), (a_s, \delta), (a_{2n+1}, 1)] \neq \emptyset. \quad (3)$$

Consider a path  $\gamma : [0, 1] \rightarrow \mathbf{P}_{2n}$  such that

$$\begin{aligned} \gamma(0) &= P_4(x), \\ \gamma(1) &= (x + \varepsilon)^{2n-1} (x + \varepsilon^{1-2n}), \quad \text{with} \\ \gamma(t) &\in \mathbf{SP}_{2n}^{\phi}[(a_0, 1), (a_{2n}, 1)] \quad \forall t \in (0, 1]. \end{aligned}$$

When  $\varepsilon \rightarrow 0$  then the coefficient of  $x^i$  for  $i = 1, \dots, 2n - 1$  in the polynomial  $(x + \varepsilon)^{2n-1} (x + \varepsilon^{1-2n})$  tends to  $\infty$ . By a continuity argument we conclude that (3) is true. And then the first part of Theorem 4.1 is proved.

(ii) Case  $d = 1$ : By the definition of  $P_4(x)$  and applying Corollary 3.1 we have that

$$\begin{aligned} P_4(x) &= \sum_{i=0}^{2n+1} b_i x^{2n+1-i} \\ &= \left( x + \frac{1}{r^{2n}} \right) (x^2 + \beta r x + r^2)^n \\ &= \left( x + \frac{1}{r^{2n}} \right) \sum_{i=0}^{2n} r^i C_{2n,i}^{\beta} x^{2n-i}. \end{aligned}$$

Therefore

$$b_s = r^s C_{2n,s}^{\beta} + \frac{C_{2n,s-1}^{\beta}}{r^{2n+1-s}}.$$

By making calculations we can see that the minimum of  $b_s$  is reached when

$$r = \left( \frac{(2n+1-s)C_{2n,s-1}^{\beta}}{sC_{2n,s}^{\beta}} \right)^{\frac{1}{2n+1}}$$

Substituting we get that

$$b_s \geq (2n+1) \left( \frac{C_{2n,s-1}^\beta}{s} \right)^{\frac{s}{2n+1}} \left( \frac{C_{2n,s}^\beta}{2n+1-s} \right)^{1-\frac{s}{2n+1}}.$$

In **(B)** we have showed that  $P(x) = \sum_{i=0}^{2n+1} a_i x^{2n+1-i} \geq P_4(x)$ , then

$$a_s \geq b_s \geq (2n+1) \left( \frac{C_{2n,s-1}^\beta}{s} \right)^{\frac{s}{2n+1}} \left( \frac{C_{2n,s}^\beta}{2n+1-s} \right)^{1-\frac{s}{2n+1}}.$$

By the argument given in **(B)** (ii) we conclude that the inequality is strict.

A similar continuity argument to that given for the case  $d = 0$  can be used now in order to show that if

$$\delta > (2n+1) \left( \frac{C_{2n,s-1}^\beta}{s} \right)^{\frac{s}{2n+1}} \left( \frac{C_{2n,s}^\beta}{2n+1-s} \right)^{1-\frac{s}{2n+1}}$$

then

$$\mathbf{SP}_{2n+1}^\phi[(a_0, 1), (a_s, \delta), (a_{2n+1}, 1)] \neq \emptyset.$$

And the proof is complete.  $\square$

## 7. Stable polynomials with margin $\varepsilon$

In practical settings perturbations are usual. Therefore, in order to control the robustness of the stability of all the elements of a family of polynomials it is necessary to assure that a small perturbation in the coefficients does not imply that some of the polynomials of the family become unstable. We will manage the situation by leaving a security margin.

A real polynomial is said to be *stable with margin*  $\varepsilon > 0$  if and only if the real part of all its roots is lesser than  $-\varepsilon$ . We will denote the set of all polynomials of degree  $n$  which are stable with margin  $\varepsilon$  by  $\mathbf{SP}_{n,\varepsilon}$ . As before, we consider only the set

$$\mathbf{SP}_{n,\varepsilon}[(a_0, 1), (a_n, 1)] = \mathbf{P}_n[(a_0, 1), (a_n, 1)] \cap \mathbf{SP}_{n,\varepsilon}.$$

Note that if  $\varepsilon \geq 1$  then all roots of a stable polynomial with margin  $\varepsilon$  have modulus greater than 1, which implies that  $\mathbf{SP}_{n,\varepsilon}[(a_0, 1), (a_n, 1)] = \emptyset$ .

**Theorem 7.1.**  $\mathbf{SP}_{n,\varepsilon}[(a_0, 1), (a_s, \delta), (a_n, 1)] = \emptyset$  for any  $\varepsilon, \delta$  with  $0 < \varepsilon < 1$  and

$$\delta \geq \binom{n-1}{s} \varepsilon^s + \binom{n-1}{s-1} \varepsilon^{s-n}.$$

**Proof.** Let  $P(x) \in \mathbf{SP}_{n,\varepsilon}[(a_0, 1), (a_n, 1)]$  and let

$$-t_1, \dots, -t_g \in \mathbb{R} \quad \text{and}$$

$$z_1 = r_1 e^{i\theta_1}, \quad \bar{z}_1, \dots, z_h = r_h e^{i\theta_h}, \quad \bar{z}_h \in \mathbb{C} - R$$

be the roots of  $P(x)$ . Note that  $g + 2h = n$  and  $\prod_{i=1}^g t_i \prod_{j=1}^h r_j^2 = 1$ . As the polynomial is  $\varepsilon$ -stable then  $t_i > \varepsilon$  for  $i = 1, \dots, g$ . Moreover, we can take  $r_j > 0$  and  $\frac{\pi}{2} < \theta_j < \pi$  for  $j = 1, \dots, h$ . Let  $\gamma_j = -2\cos\theta_j > 0$ . We define the following polynomials:

1.  $P_1(x) = P(x) = \prod_{i=1}^g (x + t_i) \prod_{j=1}^h (x^2 + \gamma_j r_j x + r_j^2)$ .
2.  $P_2(x) = \prod_{i=1}^g (x + t_i) \prod_{j=1}^h (x^2 + 2r_j x + r_j^2)$ .
3.  $P_3(x) = (x + \varepsilon^{1-n})(x + \varepsilon)^{n-1}$ .

Since  $\gamma_j < 2$  for all  $j = 1, \dots, h$  it follows that  $P_2(x) - P_1(x)$  has nonnegative coefficients. Let  $t_{g+2j-1} = t_{g+2j} = r_j$  for  $j = 1, \dots, h$ , then

$$P_2(x) = \prod_{i=1}^g (x + t_i) \prod_{j=1}^h (x + r_j)^2 = \prod_{i=1}^n (x + t_i).$$

Now we are going to compare the coefficients of  $P_2(x)$  and  $P_3(x)$ . By applying Lemma 3.1 and Lemma 5.1 it follows that  $P_3(x) - P_2(x)$  has nonnegative coefficients. It only remains to prove that the coefficient of  $x^{n-s}$  in the polynomial  $(x + \varepsilon^{1-n})(x + \varepsilon)^{n-1}$  is equal to

$$\binom{n-1}{s} \varepsilon^s + \binom{n-1}{s-1} \varepsilon^{s-n}$$

and this is an easy calculus.  $\square$

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